Discrete Mathematics 310 (2010) 2183–2197

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A_2-web immanants

Pavlo Pylyavskyy

Department of Mathematics, University of Michigan, Ann Arbor, MI, 48103, United States

ARTICLE INFO

Article history:
Received 24 November 2008
Received in revised form 2 March 2010
Accepted 18 April 2010
Available online 8 May 2010

Keywords:
Temperley–Lieb–Martin algebra
Immanants
Total positivity

ABSTRACT

We describe the rank 3 Temperley–Lieb–Martin algebras in terms of Kuperberg's A_2-webs. We define consistent labelings of webs and use them to describe the coefficients of decompositions into reduced webs. We introduce web immanants, inspired by Temperley–Lieb immanants of Rhoades and Skandera. We show that web immanants are positive when evaluated on totally positive matrices and describe some further properties.

1. Introduction

Temperley–Lieb algebras are quotients of Hecke algebras such that only the irreducible representations corresponding to Young shapes with at most two columns survive. Originally introduced in [23] for the study of percolation, Temperley–Lieb algebras appeared in many other contexts. In particular, Rhoades and Skandera in [16] used them to introduce Temperley–Lieb immanants, which are functions on matrices possessing certain positivity properties. In [12] those immanants, and their further properties developed in [17], were used to resolve some Schur-positivity conjectures. In [13] Temperley–Lieb pfaffinants were introduced, which can be viewed as "super" analogs of Temperley–Lieb immanants.

In this work we generalize in a different direction. Namely, we make use of multi-column generalizations of Temperley–Lieb algebras. Temperley–Lieb–Martin algebras (or TLM algebras) were introduced by Martin in [14]. Their irreducible representations correspond to Young shapes with at most k columns.

In [2] Brzeziński and Katriel gave a description of TLM algebras in terms of generators and relations. However, in order to deal with the combinatorics of TLM algebras, one desires more than that: it is natural to ask whether a diagrammatic calculus exists for TLM algebras similar to that of Kauffman diagrams for Temperley–Lieb algebras. It appears that the A_2 *spiders* (or *pivotal categories*) of Kuperberg [10] essentially provide such calculus for $k = 3$. The connection between centralizer algebras and spiders has been noticed previously, cf. [24].

The paper proceeds as follows. In Section 2 we review the presentation of TLM algebras obtained in [2]. We proceed to define a diagrammatic calculus for TLM algebras using the spider reduction rules of [10]. This allows us to introduce the web bases of TLM algebras. In Section 3 we introduce the tool of consistent labelings of webs, which allows us to describe the coefficients involved in the decomposition of reducible webs into reduced ones. In Section 4 we introduce web immanants. We show that web immanants are positive when evaluated on totally positive networks. In Section 5 we give a positive combinatorial formula for decomposing products of triples of complementary minors into web immanants. In Section 6 we relate web immanants and Temperley–Lieb immanants. In Section 7 we use the setting of weighted planar networks to give an interpretation of web immanants, thus providing a generalization of the Lindström's lemma. In Section 8 we discuss potential further directions.

E-mail addresses: pavlo@umich.edu, pylyavskyy@gmail.com.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.04.008

2. Web bases of TLM algebras

The *Hecke algebra* $H_n(q)$ is a free associative algebra over $\mathbb{C}(q)$ generated by elements $g_1, \ldots, g_{n-1}$ subject to the following relations:

$$g_i^2 = (q-1)g_i + q;$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1};$$

and

$$g_i g_j = g_j g_i$$

if $|i-j| > 1$. For a permutation $w \in S_n$ and a reduced decomposition $w = \prod s_{i_j}$, the element $g_w = \prod g_{i_j}$ does not depend on the choice of reduced decomposition. As w runs over all permutations in S_n, elements g_w form a linear basis for $H_n(q)$. Note that for $q = 1$ the Hecke algebra is the group algebra $\mathbb{C}S_n$ of the symmetric group.

The *Temperley–Lieb algebra* $TL_n(q^{1/2} + q^{-1/2})$ is a $\mathbb{C}(q^{1/2} + q^{-1/2})$-algebra generated by $e_1, \ldots, e_n$ with relations

$$e_i^2 = (q^{1/2} + q^{-1/2})e_i,$$
$$e_i e_{i+1} e_i = e_i e_{i-1} e_i = e_i,$$

and

$$e_i e_j = e_j e_i$$

for $|i-j| > 1$. Temperley–Lieb algebras are quotients of the Hecke algebras in which only the irreducible modules corresponding to shapes with at most two columns survive. The map $\theta_2 : g_i \mapsto q^{1/2} e_i - 1$ gives an algebra homomorphism.

Temperley–Lieb–Martin algebras are quotients of the Hecke algebra such that only the representations with at most k columns survive. Thus, for $k = 2$ those are exactly the Temperley–Lieb algebras. In [2] the following presentation for a Temperley–Lieb–Martin algebra TLM_n^k was given. Denote

$$[k]_q = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}.$$

For $i = 1, \ldots, k-1$ and $j = 1, \ldots, n-i$, we have generator $e_j^{(i)}$ subject to

$$(e_j^{(i)})^2 = [i+1]_q e_j^{(i)},$$
$$e_j^{(i+1)} = \frac{1}{[i]_q[i+1]_q}(e_j^{(i)} e_{j+1}^{(i)} e_j^{(i)} - e_j^{(i)}) = \frac{1}{[i]_q[i+1]_q}(e_{j+1}^{(i)} e_j^{(i)} e_{j+1}^{(i)} - e_{j+1}^{(i)}),$$

where we interpret

$$e_j^{(k)} = 0.$$

An algebra homomorphism

$$\theta_k : H_n(q) \longrightarrow \mathrm{TLM}_n^k(q^{1/2} + q^{-1/2})$$

is given by

$$\theta_k(g_i) = q^{1/2} e_i^{(1)} - 1,$$

it is shown in [2] that this is a well-defined map.

The generators of TL_n can be represented by *Kauffman diagrams*. Each diagram is a matching on $2n$ vertices arranged on opposite sides of a rectangle: n on the left and n on the right. Each e_i is represented by a single uncrossing between the ith and $i+1$st elements. The product is given by concatenation, with loops being erased while contributing a factor of $q^{1/2} + q^{-1/2}$. It is known that if w is a $(3, 2, 1)$-avoiding permutation and $w = \prod s_{i_k}$ is a reduced decomposition, then $e_w = \prod e_{i_k}$ does not depend on the choice of reduced decomposition. As w runs over the set of $(3, 2, 1)$-avoiding permutations, the e_w form a basis for TL_n. In terminology of [10] the non-crossing matchings on $2n$ vertices are exactly the A_1-*webs*.

A natural question is whether there exists a similar planar diagram presentation for TLM algebras. Such a presentation for $k = 3$ is implicit in [10]. Namely, Kuperberg considered A_2-*webs*. An A_2-web is a planar bipartite graph with some boundary vertices positioned around a Jordan curve and some inner vertices inside the region bounded by the curve. Each inner vertex has to have degree 3 and each boundary vertex has to have degree 1. In addition, an orientation on edges of the web is given that makes every vertex either a source or a sink. The six possible A_2-webs with three sources followed by three sinks on the boundary are shown in Fig. 2. Unless specified otherwise the word web will refer to A_2-web in what follows.

The *spider reduction rules* in Fig. 1 were introduced in [10]. A web is *reduced* if no reduction rule can be applied to it. It is known that every non-reduced web can be uniquely reduced to a linear combination of reduced webs using the above rules, cf. [11, Theorem 1.2], [19, Corollary 5.1].

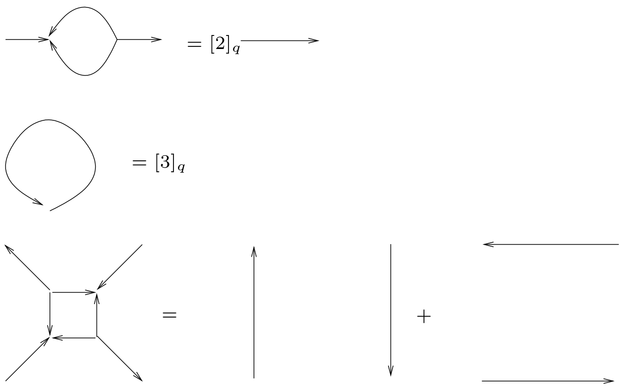

Fig. 1.

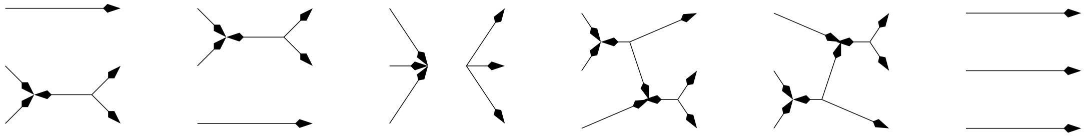

Fig. 2.

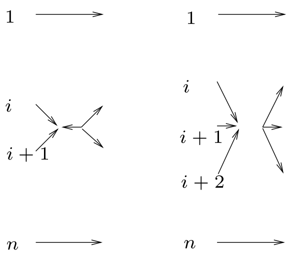

Fig. 3.

Remark 2.1. Note that the webs introduced in [10] have arbitrary boundary conditions, while we restrict our attention to the webs having n sources on the left and n sinks on the right. Note also that the rules in [10] actually differ by sign. For a reason to be evident later we prefer the positive version.

Let W_n be an algebra generated by the diagrams in Fig. 3 with the product given by concatenation and relations given by the spider reduction rules. Let

$$\eta : \mathrm{TLM}_n^3 \longrightarrow W_n$$

be given by mapping $e_i^{(1)}$ into the first type of diagram in Fig. 3, and the elements $[2]_q e_i^{(2)}$ into the second type of diagram (note the coefficient).

Theorem 2.2. *The map η is an algebra isomorphism.*

Proof. The defining relations of TLM_n^3 are easily verified inside W_n, as shown in Fig. 4. On the other hand, according to [9, Theorem 6.1] the dimension of W_n is equal to the dimension of the space of $\mathfrak{sl}_3$-invariants $\mathrm{Inv}(V_{(1)}^{\otimes n} \otimes V_{(1,1)}^{\otimes n})$, where $V_{(1)}$ and $V_{(1,1)}$ are the two fundamental representations of $\mathfrak{sl}_3$. This number is equal to the Kostka number $K_{3^n,1^n2^n}$, see for example [6, Chapter 8]. Lemma 2.3 tells us that this number is exactly the dimension of TLM_n^3, see [2]. Thus, the dimensions of W_n and TLM_n^3 are equal. We postpone the proof of injectivity until Theorem 3.3. The two facts together imply that η is an isomorphism.

Lemma 2.3. *$K_{3^n,1^n2^n}$ is equal to the number of pairs of standard Young tableaux of the same shape λ such that $|\lambda| = n$ and λ has at most three columns. This number is equal to the number of $(4, 3, 2, 1)$-avoiding permutations of size n.*

Proof. The position of 1^n in the rectangle gives a standard filling of a shape of size n and at most three columns. Each next pair of cells contributing 2 into the weight has to leave another top row of the rectangle filled; otherwise as it is easy to see,

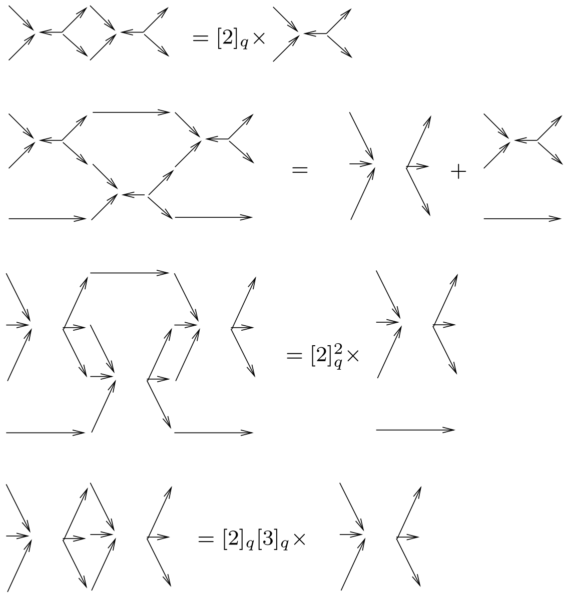

Fig. 4.

the filling cannot be completed. Cut out the top row of the remaining rectangle at each such step and see what shape inside it is currently occupied. Then, the resulting shapes form a nested sequence inside the original shape, each next one missing a square compared to the previous one. Such sequences are in bijection with standard fillings of the shape. Thus, we have produced two standard fillings of the same shape satisfying criteria above. For example the following semistandard tableau of weight $1^4 2^4$ produces the following two standard Young tableaux of the same shape:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 5 & 6 \\ \hline 4 & 7 & 7 \\ \hline 6 & 8 & 8 \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} .$$

Here the first standard tableau is read off directly from the original tableau, while the second one records the contraction of the shape:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|}\hline 4 \\ \hline 6 \\ \hline \end{array} \longrightarrow \begin{array}{|c|}\hline 6 \\ \hline \end{array}.$$

It is easy to see that the construction is reversible and gives a bijection, which completes the argument. The second statement of the lemma follows from RSK correspondence and the theory of Greene–Kleitman invariants, cf. [6, Chapters 3–4]. □

As a result of Theorem 2.2, one can define elements

$$e_D = \eta^{-1}(D)$$

of TLM_n^3 for each web D occurring in W_n. As D runs over reduced webs, the elements e_D form a *web basis* of TLM_n^3. Note that unlike in the case of Temperley–Lieb algebra, the e_D are not always monomials in the $e_i^{(j)}$. For example, in TLM_4^3 one of the webs can be expressed as $[2]_q(e_2^{(1)}e_1^{(1)}e_2^{(2)} - e_2^{(2)})$.

3. Consistent labelings

3.1. Definition and statistic

Denote by $\mathfrak{M}_n$ the set of webs D of size n. Each edge $e \in D$ has two sides, which we denote e_+ and e_-, so that every edge is directed from its positive to its negative side. In other words, since a web is bipartite, we distinguish the parts of edges adjacent to sources from the parts of edges adjacent to sinks. A *consistent labeling* of D is an assignment of a label

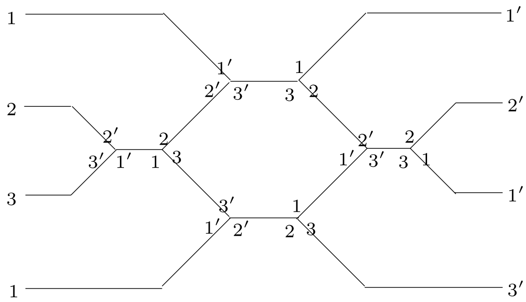

Fig. 5.

$f : e_\pm \mapsto 1, 2, 3, 1', 2', 3'$ to each side of each edge so that the following conditions hold:

(1) positive sides are labeled with 1, 2, 3, and negative sides are labeled with $1', 2', 3'$;
(2) if e_+ is labeled with i then e_- is labeled with i';
(3) the labels adjacent to the same vertex are distinct, i.e., the sides of edges adjacent to every degree 3 vertex in D are labeled either with 1, 2, 3 or with $1', 2', 3'$.

The labels adjacent to boundary vertices of D are called *boundary labels*. The restriction g of a labeling f to the boundary is called a *boundary labeling*. Let L_D denote the set of all consistent labelings of D, and $L_{D,g}$ denote the set of all consistent labelings with a prescribed boundary labeling g.

An example of a consistent labeling is given in Fig. 5. For this web and this boundary labeling, there exists only one consistent labeling, i.e., $|L_{D,g}| = 1$.

Let a *singularity* of a consistent labeling be one of the following:

(1) a degree 3 vertex in D;
(2) a point on an edge of D whose tangent line is vertical.

Readjusting the embedding of D one can clearly make its edges non-vertical, and can make no two singularities lie on one vertical line.

Let v be a singularity of the first kind. Let p, q, r be labels adjacent to v, so they are either 1, 2, 3 or $1', 2', 3'$. Define order on the labels as follows: $1 < 2 < 3$ and $3' < 2' < 1'$. Let l_v be the vertical line passing through v. For an unordered pair of labels (p, q) adjacent to v define

$$\alpha_v(p,q) = \begin{cases} -1 & \text{if } p < q,\ p \text{ and } q \text{ both lie to the left of } l_v \text{ and } p \text{ is above } q; \\ -1 & \text{if } p < q,\ p \text{ and } q \text{ both lie to the right of } l_v \text{ and } p \text{ is below } q; \\ 1 & \text{if } p > q,\ p \text{ and } q \text{ both lie to the left of } l_v \text{ and } p \text{ is above } q; \\ 1 & \text{if } p > q,\ p \text{ and } q \text{ both lie to the right of } l_v \text{ and } p \text{ is below } q; \\ 0 & \text{if } p \text{ and } q \text{ lie on different sides of } l_v. \end{cases}$$

Let

$$\alpha(v) = \alpha_v(p,q) + \alpha_v(p,r) + \alpha_v(q,r)$$

be the sum taken over all pairs of labels adjacent to v.

Let v be now a singularity of the second kind, and again let l_v be the vertical line passing through v. Recall that each edge of D is oriented from some i to i'. Assume that at v line l_v is tangent to the edge labeled i at the beginning, i' at the end. Let

$$\alpha(v) = \begin{cases} 4-2i & \text{if edge is oriented down around } v \text{ and touches } l_v \text{ from the left;} \\ 2i-4 & \text{if edge is oriented down around } v \text{ and touches } l_v \text{ from the right;} \\ 4-2i & \text{if edge is oriented up around } v \text{ and touches } l_v \text{ from the right;} \\ 2i-4 & \text{if edge is oriented up around } v \text{ and touches } l_v \text{ from the left.} \end{cases}$$

Now for a consistent labeling f of D define

$$\alpha(f) = \prod_v q^{\frac{\alpha(v)}{4}},$$

where the product is taken over all singularities of a particular embedding of D.

Example 3.1. The leftmost singularity v shown in Fig. 5 has $2'$ and $3'$ to the left of l_v, $2'$ above $3'$, and $1'$ to the right of l_v. Then, $\alpha(2', 3') = 1$ while $\alpha(1', 2') = \alpha(1', 3') = 0$, which results in $\alpha(v) = 1$. For this embedding of the web, there are no singularities of the second kind and for this particular labeling f we have $\alpha(f) = q^{\frac{1+1+1-1+1+1-1-1}{4}} = q^{1/2}$, where the summands in the exponent correspond to singularities on Fig. 5 left to right.

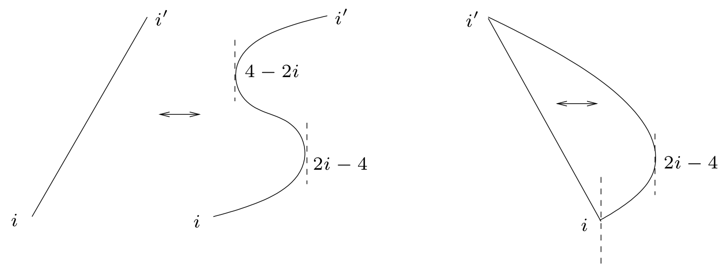

Fig. 6.

The following is the key property of the statistic α.

Lemma 3.2. *$\alpha(f)$ does not depend on the particular embedding of web D.*

Proof. It is easy to see that any two embeddings of a given web can be deformed into one another by a sequence of the moves of the following two types:

(1) bending or unbending edges without changing the direction at the ends, cf. first part of Fig. 6;
(2) changing the direction of one of the edges at its end, with simultaneous creation or removal of a singularity next to it, cf. second part of Fig. 6.

Here in Fig. 6 vertical lines are drawn to show the singularities. The labels on ends of the edges are also shown, as well as values of $\alpha(v)$ for each singularity v.

Consider the first type of move. The two singularities created by such bending cancel out, contributing total of $q^{\frac{2i-4}{4}}q^{\frac{4-2i}{4}} = 1$ into $\alpha(f)$. Other cases with different orientation of the edge being bent are similar.

Consider now the second type of move. As shown in second part of Fig. 6, an edge changes its side with respect to l_v, where v is a singularity, and because of that a new singularity is created. For each pair of labels $j < i$ adjacent to v, the value $\alpha(i,j)$ becomes one less than it used to be: it either used to be 1 and became 0, or it used to be 0 and became -1. Similarly for each $j > i$ the value of $\alpha(i,j)$ is one more than it used to be. That results in the total factor of $q^{-\frac{i-1}{4}}q^{\frac{3-i}{4}}$. This however cancels out with the new factor $q^{\frac{2i-4}{4}}$ coming from the new singularity. Other cases given by different orientation or different distribution of the edges around the singularity v are similar. $\square$

Denote

$$|L_{D,g}|_q = \sum_{f \in L_{D,g}} \alpha(f),$$

we refer to $|L|_q$ as to *q-size* of L. Note that when $q = 1$, the q-size $|L_{D,g}|_q = |L_{D,g}|$ is just the number of elements in $L_{D,g}$.

3.2. Properties

Recall that all webs in $\mathfrak{M}_n$ have $2n$ boundary vertices: n on the left and n on the right. Let G_n be the set of all possible boundary labelings g of the $2n$ boundary vertices, and consider the vector space R_n over $\mathbb{C}(q^{1/2} + q^{-1/2})$ spanned by the abstract variables r_g, $g \in G$. We define an algebra structure on R_n as follows: $r_{g_1}r_{g_2}$ is equal to

(1) r_g, where g is the boundary labeling obtained by combining the left half of g_1 and right half of g_2, if the right half of g_1 is obtained from the left part of g_2 via map $i \mapsto i'$;
(2) 0 otherwise.

It is not hard to see that this product turns R_n into an associative algebra with unity. Consider the map $\kappa : W_n \longrightarrow R_n$ defined by $\kappa : D \mapsto \sum_{g \in G_n} |L_{D,g}|_q r_g$.

Theorem 3.3. *The map κ is an injective algebra homomorphism, and so is the map η of Theorem 2.2.*

Proof. The concatenation product in W_n is clearly compatible with the product structure of R_n. Thus, in order to check that κ is an algebra homomorphism, we need to verify that the defining relations of W_n are satisfied in R_n. In particular it is enough to check that spider reduction rules are compatible with κ.

The first two reduction rules from Fig. 1 are easy to verify. For example, a closed loop produces two singularities. If the loop is oriented for example clockwise, and labeled by i and i', then the two singularities contribute the factor $q^{\frac{4-2i}{4}}$ each. Thus, as i ranges through the possible values of 1, 2, 3, the total factor contributed is $q^{-1} + 1 + q = [3]_q$ just as it should be according to spider rules.

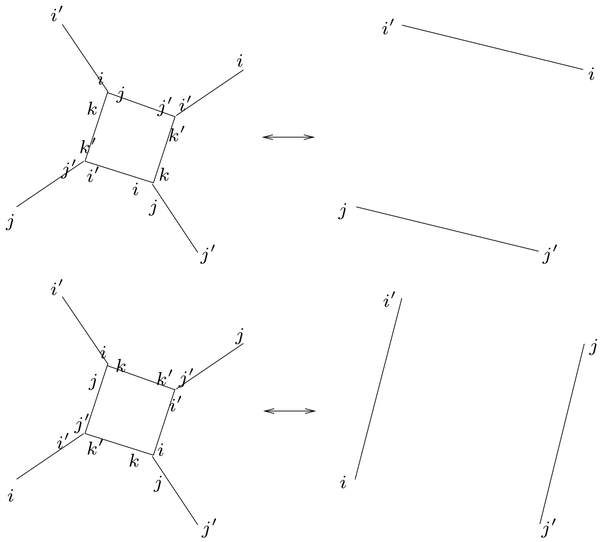

Fig. 7.

Let us therefore deal with the third rule. There is only one way to label the sides in the square configuration from the third rule. Namely, the boundary must contain a pair of labels i and i' and another pair of j and j'. The only distinction comes from the relative position of those labels, the two possibilities shown in the Fig. 7.

One can see that in both of the cases shown in Fig. 7, every consistent labeling of some web containing the square configuration is in bijection with a consistent labeling of exactly one of the two possible resolutions. Moreover, the statistic $\alpha(f)$ is preserved. For example in the upper case, the singularity with i, k on the left cancels out with singularity with k', i' on the right while the singularity with k, j on the right cancels out with singularity with j', k' on the left. The other case is similar.

Now we want to deduce the injectivity, i.e., that W_n can be realized inside R_n. For each $(4, 3, 2, 1)$-avoiding permutation $w \in S_n$ pick a reduced decomposition $\bar{w} = \prod s_{i_j}$ and consider the monomial $e_{\bar{w}} = \prod e^{(1)}_{i_j}$ in TLM_n^3. We use a triangularity argument to show that images $\kappa(\eta(e_{\bar{w}}))$, as w varies over all $(4, 3, 2, 1)$-avoiding permutations, are linearly independent.

Let $D_{\bar{w}}$ be the web obtained by concatenation of webs of generators $e^{(1)}_{i_j}$ according to $\bar{w}$. It is a well-known result (going back to Erdös) that every $(4, 3, 2, 1)$-avoiding permutation can be partitioned into three increasing subsequences. For a given w pick one such partitioning and label the boundary of $D_{\bar{w}}$ according to this partitioning in the following sense: kth vertex from the top on the left should be labeled i if and only if $w(k)$th vertex from the top on the right is labeled i', and vertices belonging to the same part have the same label. Denote by g_w the resulting boundary labeling. For example, if $n = 4$, $w = (1, 4, 3, 2)$ and the partitioning is $(1, 4) \cup (2) \cup (3)$, label the sources with $1, 2, 3, 1$ and the sinks with $1', 1', 3', 2'$ top to bottom.

Note that the boundary labeling vector r_{g_w} occurs in the decomposition of $\kappa(\eta(e_{\bar{w}}))$. To see this fact, label each diagram D_{s_i} (constituting part of $D_{\bar{w}}$) so that the output labels are transposed input labels. Since $\bar{w}$ is a reduced decomposition, and since in g_w entries having the same label increase, the resulting labeling is consistent. On the other hand, any permutation that produces g_w can be written as combination of w and some further transpositions between entries with the same labels. The length of the resulting permutation is bigger than that of w. Such a permutation cannot possibly be achieved by skipping some steps in $\bar{w}$. Therefore, r_{g_w} occurs in the decomposition of $\kappa(\eta(e_{\bar{w}}))$ with a non-zero coefficient (in fact with coefficient 1).

Now take any linear extension of the Bruhat order. Then, the boundary labeling vector r_{g_w} cannot occur in the decomposition of any $\kappa(\eta(e_{\bar{v}}))$ for $v < w$ in the chosen order. This essentially was proven above, given the sub-word characterization of the Bruhat order. Therefore, the $\kappa(\eta(e_{\bar{w}}))$ are indeed linearly independent and the dimension of the image of TLM_n^3 in R_n is equal to the number of $(4, 3, 2, 1)$-avoiding permutations in S_n. As we already know this is exactly the dimension of TLM_n^3 and the injectivity of both κ and η follows. $\square$

Let $D \in \mathfrak{M}_n$ be a non-reduced web, and let $D = \sum c_i D_i$ be the unique expression for D as a linear combination of reduced webs D_i. Let us consider the process of reduction of D using the spider reduction rules from Fig. 1, and write it schematically as a tree in the following way. Each time we apply the first or the second rule, we add the resulting web to the tree as a single child, with an extra coefficient coming from the rule applied. We write this coefficient on the edge connecting to the newly added web. When we apply the third spider reduction rule however, we write the resulting two webs as children and go on with reducing each of them separately from that point on. One can think of the two newly created edges as having coefficients 1 on them. The process ends at some point, producing a binary tree with reduced webs in its leaves. Each such

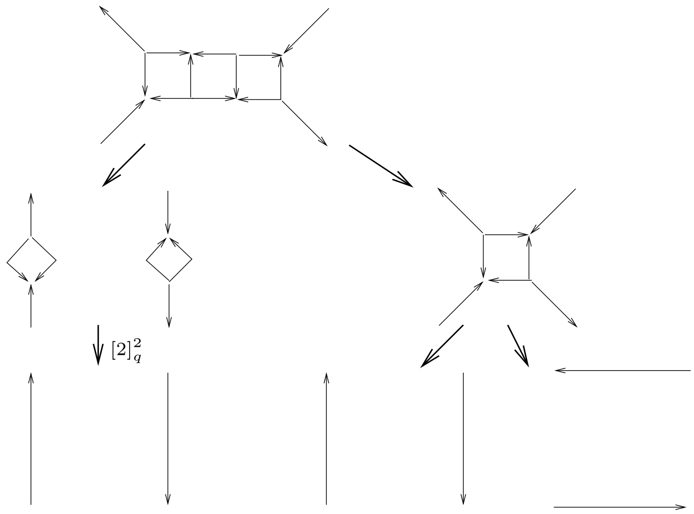

Fig. 8.

reduced web inherits a coefficient during the process, which can be computed by multiplying coefficients on edges along the unique path from the root of the tree to this particular leaf. If for a leaf l we denote c_l the coefficients computed in this manner, then $c_i = \sum_l c_l$, where the sum is taken over all leaves l such that the reduced web in l is D_i. The Fig. 8 illustrates a possible tree (a fragment of the whole web is shown). In this example, the two reduced webs located in the leaves of the resulting tree get coefficients $c_1 = [2]_q^2 + 1$ and $c_2 = 1$.

Let us now start with a consistent labeling of D. From the proof of Theorem 3.3 we know that when each of the spider reduction rules is applied, we get a map from the consistent labelings of the original web to the consistent labelings of the resulting web. Furthermore, at each branching point, corresponding to an application of the third rule, the current labeling dictates into which of the two branches we go, cf. Fig. 7. Thus, we can define the *type* of an original labeling f as the reduced web D_i we end up with. Note that the type of a labeling a priori might depend on the choice of spider reduction steps. It seems likely that it is actually independent of the choices made; however, it is not necessary for the further argument. From now on we assume that for every possible web, one possible branching sequence of reduction steps is chosen. Or in other words, to every web a fixed tree is associated as above.

Let g be a boundary labeling for D. Let $L_{D,D_i,g}$ denote the set of consistent labelings of D with boundary g and of type D_i.

Theorem 3.4. *Let $D \in \mathfrak{M}_n$ be a web, and let $D = \sum c_i D_i$ be the unique expression for D as a linear combination of reduced webs D_i. Then, each coefficient c_i satisfies*

$$c_i = \begin{cases} \dfrac{|L_{D,D_i,g}|_q}{|L_{D_i,g}|_q} & \text{if } |L_{D_i,g}| > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Choose a particular sequence of reductions producing a binary tree as above. The spider reduction rules provide a surjection from the consistent labelings of D with boundary g and of type D_i onto the consistent labelings of the D_i-leaves. The relative q-size of the fiber of each letter is exactly the coefficient that appears by applying the spider rules when we descend into that particular leaf. Therefore, the set $L_{D,D_i,g}$ of consistent labelings of D with boundary g of type D_i gets partitioned into the union of sets with q-size $c_{l,i}|L_{D_i,g}|_q$ as l runs over type D_i leaves and $c_{l,i}$ is the coefficient created when descending into leaf l. Since by definition $c_i = \sum_l c_{l,i}$, we conclude the needed statement. $\square$

The rest of the paper proceeds with $q = 1$.

4. Web immanants and total positivity

For a function $f : S_n \longrightarrow \mathbb{C}$ and an $n \times n$ matrix X an *immanant* $\mathrm{Imm}_f(X)$ is defined by

$$\mathrm{Imm}_f(X) = \sum_{w \in S_n} f(w) x_{1,w(1)}, \ldots, x_{n,w(n)}.$$

We define web immanants by analogy with the *Temperley–Lieb immanants* of Rhoades and Skandera [16].

The map θ_k defined in Section 2 for $k = 3$ and $q = 1$ specializes to the map

$$\theta_3 : \mathbb{C}S_n \longrightarrow \mathrm{TLM}_n^3(2)$$

given by

$$\theta_3(g_i) = e_i^{(1)} - 1.$$

For each reduced web $D \in \mathfrak{M}_n$ and $w \in S_n$, let $f_D(w)$ be the coefficient of e_D in the image $\theta_3(w)$. Then, the immanants

$$\mathrm{Imm}_D(X) = \mathrm{Imm}_{f_D}(X) = \sum_{w \in S_n} f_D(w) x_{1,w(1)}, \ldots, x_{n,w(n)}$$

are called *web immanants*.

Following [17] let $z_{[i,j]}$ denote the sum of all elements of the parabolic subgroup of S_n generated by $s_i, \ldots, s_{j-1}$. We will make use of the following lemma.

Lemma 4.1. (1) $\theta_3(z_{[i,i+1]}) = \theta_3(s_i + 1) = e_i^{(1)}$;
(2) $\theta_3(z_{[i,i+2]}) = 2e_i^{(2)}$;
(3) $\theta_3(z_{[i,i+k]}) = 0$ *for* $k > 2$.

Proof. The first part is clear from the definition. For the second part, one checks that $(s_i+1)(s_{i+1}+1)(s_i+1) - (s_i+1) = z_{[i,i+2]}$. Finally, for the third part one can check that $z_{[i,i+3]} = 6e_i^{(3)} = 0$ and for any $k > 3$, $z_{[i,i+3]}$ is a factor of $z_{[i,i+k]}$ in $\mathbb{C}S_n$. $\square$

Now we are ready to consider the properties of web immanants.

Recall that a real matrix is *totally nonnegative* if the determinants of all its minors are nonnegative, see for example [5] and references there. We define an immanant to be totally nonnegative if, when applied to any totally nonnegative matrix, it produces a nonnegative number. For example, by definition the determinant is totally nonnegative. The following theorem is similar to [16, Theorem 3.1] and [13, Proposition 32].

Theorem 4.2. *Web immanants are totally nonnegative.*

The proof resembles the proof of [17, Proposition 2]. In particular we will need the following lemma.

Lemma 4.3 ([16, Lemma 2.5], [21, Theorem 2.1]). *Given a totally nonnegative matrix X, it is possible to choose a set Z of elements of $\mathbb{C}S_n$ of the form $z = \prod z_{[i_k,j_k]}$ and nonnegative numbers c_z, $z \in Z$ so that*

$$\sum_{w \in S_n} x_{1,w(1)}, \ldots, x_{n,w(n)} w = \sum_{z \in Z} c_z z.$$

With this we are ready to prove the theorem.

Proof. Let X be a totally nonnegative matrix and let $\sum c_z z$ be the expression as in Lemma 4.3. Then,

$$\mathrm{Imm}_D(X) = \sum c_z f_D(z).$$

Note however that by Lemma 4.1 $\theta_3(z)$ is a monomial in the $e_i^{(j)}$. According to spider reduction rules each such monomial is a nonnegative combination of the e_D. Therefore, $f_D(z) \geq 0$ for all $z \in Z$ and $\sum c_z f_D(z) \geq 0$. $\square$

5. Complementary minors

For two subsets $I, J \subset [n]$ of the same cardinality, let $\Delta_{I,J}(X)$ denote the minor of an $n \times n$ matrix X with row set I and column set J. A set of minors is called *complimentary* if each row and column index participates in exactly one of the minors.

Let (I_1, J_1), (I_2, J_2) and (I_3, J_3) be a triple of complementary minors. Define the boundary labeling g by the following rule: I_1, I_2, I_3 prescribe which of the source vertices are adjacent to edge sides labeled by 1s, 2s and 3s correspondingly, while J_1, J_2, J_3 prescribe which of the sink vertices are adjacent to edge sides labeled by $1'$s, $2'$s and $3'$s correspondingly. The following theorem is similar to [16, Proposition 4.3] and [13, Theorem 7].

Theorem 5.1. *We have*

$$\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X)\Delta_{I_3,J_3}(X) = \sum |L_{D_i,g}| \mathrm{Imm}_{D_i}(X),$$

where the sum is taken over all reduced webs $D_i \in \mathfrak{M}_n$.

Example 5.2. The fact that $|L_{D,g}| = 1$ in the example in Fig. 5 means that when the product of minors

$$\begin{vmatrix} x_{1,1} & x_{1,3} \\ x_{4,1} & x_{4,3} \end{vmatrix} \cdot x_{2,2} \cdot x_{3,4}$$

is decomposed into web immanants the coefficient of Imm_D for this particular reduced web D is equal to 1.

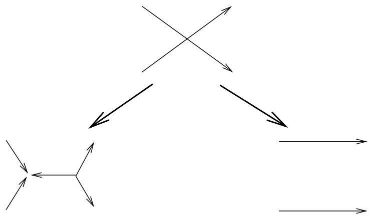

Fig. 9.

Thus, we have a positive combinatorial rule for expressing the products of triples of complementary minors in terms of web immanants. Note that unlike in the Temperley–Lieb case, the expression is not necessarily multiplicity free, since it can happen that $|L_{D,g}| > 1$.

Proof. The general idea of the proof is to show that coefficients of a particular monomial on the left and on the right coincide. Let w be a permutation and let $\bar{w} = \prod s_{i_j}$ be a reduced decomposition for w. We wish to express $\theta_3(w) = \prod (e_{i_j}^{(1)} - 1)$ as a linear combination of webs. If we just expand by linearity but do not perform any cancellation, $\prod (e_{i_j}^{(1)} - 1)$ is an alternating sum $\sum c_D e_D$ of the e_D, where each web D is a concatenation of webs corresponding to e_{i_j}-s (i.e., the first kind of webs from Fig. 3) and 1-s (i.e., the identity webs).

Alternatively, one can get the same result by starting with the wiring diagram of w corresponding to the reduced decomposition $\bar{w}$. Then, each web D in the above expression is obtained from this wiring diagram by uncrossing all crossings in one of the two ways, as shown on Fig. 9. We refer to them as *vertical* and *horizontal* uncrossings, the vertical ones correspond to choosing e_{i_j} and the horizontal ones correspond to choosing -1 in parentheses. In particular, the coefficients c_D are equal to ± 1 and the sign is determined by contribution of a factor -1 each time we uncross horizontally.

Now we wish to extract from $\theta_3(w) = \sum c_D e_D$ the coefficients of reduced webs, since those are the constants used to define web immanants. By Theorem 3.4 we know that for a particular reduced web D_i such that $|L_{D_i,g}| \neq 0$ the coefficient of e_{D_i} in e_D is equal to $\frac{|L_{D,D_i,g}|}{|L_{D_i,g}|}$. Here, g is chosen to be the boundary labeling determined by the triple of complementary minors we have, as described before the statement of the Theorem 5.1. Therefore, the coefficient in $\theta_3(w)$ of a particular e_{D_i} such that $|L_{D_i,g}| \neq 0$, is equal to

$$\sum_D c_D \frac{|L_{D,D_i,g}|}{|L_{D_i,g}|},$$

where the sum is taken over all D-s involved in the expression for $\theta_3(w)$.

Now we are interested in the coefficient of $x_{1,w(1)}, \ldots, x_{n,w(n)}$ in the right part of the equality to be proved, i.e. in $\sum |L_{D_i,g}| \mathrm{Imm}_{D_i}(X)$. By the definition of web immanants and the remark above, it is equal to

$$\sum_{D,D_i} c_D \frac{|L_{D,D_i,g}|}{|L_{D_i,g}|} |L_{D_i,g}| = \sum_{D,D_i} c_D |L_{D,D_i,g}|,$$

where the sum is taken over all D-s appearing in the expression for $\theta_3(w)$ and all reduced webs D_i such that $|L_{D_i,g}| \neq 0$. Note however that if $|L_{D_i,g}| = 0$, then $|L_{D,D_i,g}| = 0$, and thus the sum in the formula can actually be taken over *all* reduced D_i. Next, for a given web D,

$$\sum_{D_i} |L_{D,D_i,g}| = |L_{D,g}|,$$

since every consistent labeling of D has one of the reduced webs D_i as its type. Therefore, the coefficient of $x_{1,w(1)}, \ldots, x_{n,w(n)}$ in $\sum |L_{D_i,g}| \mathrm{Imm}_{D_i}(X)$ is equal to

$$\sum_{D,D_i} c_D |L_{D,D_i,g}| = \sum_D c_D \sum_{D_i} |L_{D,D_i,g}| = \sum_D c_D |L_{D,g}|,$$

where the sum is as usually over all webs D involved in the expression $\theta_3(w) = \sum c_D e_D$ described in the first paragraph of the proof.

The expression $\sum_D c_D |L_{D,g}|$ is an alternating sum of quantities $L_{D,g}$, each of which is the number of consistent labelings of a web D that are compatible with the boundary labeling g. We wish to greatly simplify this expression by constructing an involution which would pair a consistent labeling of one of the webs with a consistent labeling of another web so that the signs of the two webs in the latter expression are opposite. After that we would only need to count labelings that were not paired.

Note that there are two essentially different ways to label consistently a vertical uncrossing, as shown on Fig. 10. We refer to the first way as *unstable*, and to the second way as *stable*. Similarly, we refer to a horizontal uncrossing as *stable* if

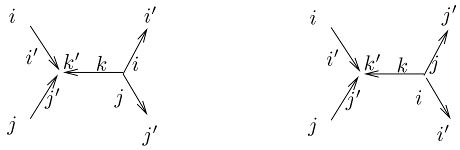

Fig. 10.

the labels on its two edges are equal, and *unstable* otherwise. Observe that every unstable uncrossing can be changed into a unique unstable uncrossing of the opposite kind, i.e., vertical to horizontal and horizontal to vertical.

Choose a planar embedding of the original wiring diagram of w which does not have two crossings on the same vertical line. We define an involution on the set of all labelings of all possible uncrossed diagrams D entering $\sum c_D e_D$ as follows. Choose the leftmost unstable uncrossing. Swap the type of uncrossing, changing the labeling correspondingly. It is easy to see that this gives an involution.

Note that the two webs carrying the original and the resulting labelings enter $\sum c_D e_D$ with distinct signs, since one contains one more horizontal uncrossing than the other. Therefore, corresponding terms in $\sum c_D |L_{D,g}|$ cancel out. The only terms that remain are the ones with all uncrossings stable. There is at most one such uncrossing/labeling, and it must have the following properties:

(1) if source m is adjacent to label i then sink $w(m)$ is adjacent to label i' (here we say that w *agrees* with g);
(2) all wires originating in sources with the same label uncross horizontally, all wires originating in sources with different labels uncross vertically.

Then, if the number of horizontal uncrossings is l, the resulting coefficient is given by

$$\sum_D c_D |L_{D,g}| = \begin{cases} (-1)^l & \text{if } w \text{ agrees with } g; \\ 0 & \text{otherwise.} \end{cases}$$

This number is exactly the coefficient of $x_{1,w(1)}, \ldots, x_{n,w(n)}$ in $\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X)\Delta_{I_3,J_3}(X)$. □

Theorem 5.3. *Web immanants form a basis for the vector space generated by triples of complementary minors.*

Proof. According to [3] the dimension of the space generated by products of triples of complementary minors is equal to the number of pairs of standard Young tableaux of the same shape with at most three rows. By the properties of Robinson–Schensted–Knuth insertion algorithm, cf. [6], this number is exactly the number of (4, 3, 2, 1)-avoiding permutations. The statement then follows from Theorem 5.1. □

6. Relation to Temperley–Lieb immanants

For a (3, 2, 1)-avoiding permutation w and a permutation v, let $f_w(v)$ be the coefficient of e_w in $\theta_2(v)$. In [16] the *Temperley–Lieb immanants* were defined as

$$\mathrm{Imm}_w^{TL}(X) = \sum_{v \in S_n} f_w(v) x_{1,v(1)}, \ldots, x_{n,v(n)}.$$

Recall that each (3, 2, 1)-avoiding permutation w corresponds to a non-crossing matching on $2n$ vertices, which is the Kauffman diagram for the basis element e_w of the Temperley–Lieb algebra. By abuse of notation we denote this matching also as w. Recall from Section 2 that A_1-webs are exactly the non-crossing matchings on $2n$ vertices. For technical reasons to be evident soon we want to place an extra vertex on those edges of an A_1-web that have both ends on the same side: either among the left n vertices or among the right n vertices. An example is given on Fig. 11. A *consistent labeling* of an A_1 web is an assignment of labels $1, 1', 2, 2'$ to sides of edges so that

(1) every edge is labeled by i, i';
(2) every internal vertex is adjacent either to 1, 2 or to $1', 2'$;
(3) the left n boundary vertices are labeled by 1, 2, and the right n boundary vertices are labeled by $1', 2'$.

It is easy to see that consistent labelings are possible because of the extra vertices we dropped.

Let (I_1, J_1) and (I_2, J_2) be a pair of complementary minors, and let g be the corresponding boundary labeling with $1, 1', 2, 2'$. Let $M_{w,g}$ denote the set of consistent labelings of w that are compatible with g. It is easy to see that $M_{w,g}$ is either empty or contains exactly one labeling. The following property of Temperley–Lieb immanants was proved in [16, Proposition 4.3].

Theorem 6.1.

$$\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X) = \sum_w |M_{w,g}| \mathrm{Imm}_w^{TL}(X).$$

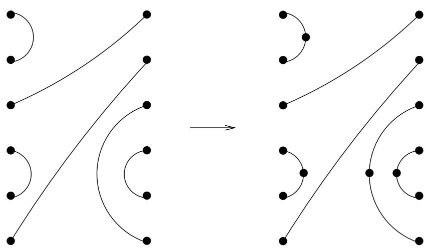

Fig. 11.

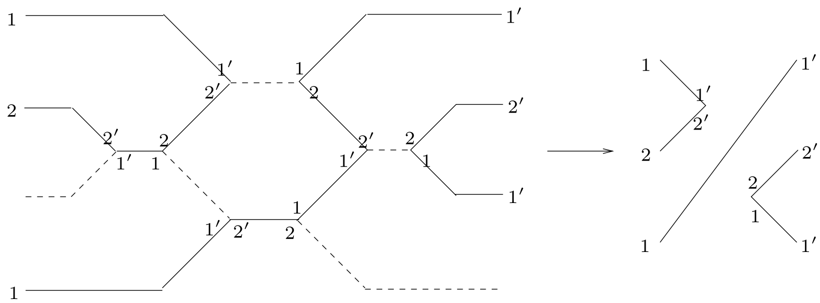

Fig. 12.

Now let (I, J) and (I_3, J_3) be a pair of complementary minors. Let $\mathrm{Imm}^{TL}_w(X')$ be a Temperley–Lieb immanant of the submatrix X' of X with row set I and column set J. Since $\mathrm{Imm}^{TL}_w(X')$ lies in the subspace of products of pairs of complementary minors of X', the product $\mathrm{Imm}^{TL}_w(X')\Delta_{I_3,J_3}(X)$ lies in the subspace of products of triples of complementary minors of X. Therefore, it must be expressible in terms of web immanants:

$$\mathrm{Imm}^{TL}_w(X')\Delta_{I_3,J_3}(X) = \sum_D a^D_{w,I_3,J_3}\mathrm{Imm}_D(X).$$

There exists a *forgetful map* from consistent labelings of webs to consistent labelings of A_1-webs, given by deleting all edges labeled with $(3, 3')$ and ignoring the loops, should any appear. See Example 6.3. Let w be an A_1-web, we identify vertices of w with vertices of D given by sets I_1, I_2, J_1, J_2. Let us denote by $L_{D,g,w}$ the set of consistent labelings of a web D compatible with the boundary labeling g and mapped by the forgetful map to a consistent labeling of w. Let $\tilde{g}$ range over boundary labelings with positions of 3s and 3′s given by (I_3, J_3) and such that $M_{w,g}$ is non-empty. The following theorem gives an interpretation of the transition coefficients a^D_{w,I_3,J_3}.

Theorem 6.2. *The size of $L_{D,\tilde{g},w}$ does not depend on the particular choice of $\tilde{g}$ and we have $a^D_{w,I_3,J_3} = |L_{D,\tilde{g},w}|$.*

Example 6.3. For the reduced web on Fig. 5, the shown labeling is the only one having $I_3 = \{3\}, J_3 = \{4\}$ and mapped by the forgetful map to the A_1-web corresponding to $w = (2, 3, 1)$, cf. Fig. 12.

Thus, the coefficient of $\mathrm{Imm}_D(X)$ in $\mathrm{Imm}^{TL}_w(X') \cdot x_{3,4}$ is 1.

Proof. Note that any two consistent labelings of w can be obtained one from the other by several applications of the following procedure. Choose a chain of edges of w connecting two boundary vertices, and change all labels along those edges so that 1 changes into 2 and vice versa, while $1'$ changes into $2'$ and vice versa. Iterated application of this procedure in fact gives a family of bijective maps between consistent labelings of w with different boundary labelings g, as g ranges over boundary labelings such that $M_{w,g}$ is non-empty. Those bijections can be lifted to elements of the $L_{D,\tilde{g},w}$ in the following way. Each chain of edges of w connecting two boundary vertices comes via a forgetful map from a chain of $1, 2, 1', 2'$-labeled edges of the A_2-web D. Perform the change of labels on those edges of D as prescribed above. It is clear that iteration of such maps is a bijection, this time on labelings of D. An example is given in Fig. 13, where a change of labeling of an edge in w is lifted to a change of labelings of a chain of edges in D. This shows independence of $L_{D,\tilde{g},w}$ on the choice of $\tilde{g}$.

Now define alternative immanants

$$\mathrm{Imm}^a_w(X) = \sum_D |L_{D,\tilde{g},w}|\mathrm{Imm}_D(X),$$

where $\tilde{g}$ is one of the boundary labelings such that $M_{w,g}$ is non-empty. Let (I_1, J_1) and (I_2, J_2) be a pair of complementary minors of X' and let g be the corresponding boundary labeling. We know from Theorem 5.1 that

$$\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X)\Delta_{I_3,J_3}(X) = \sum_D |L_{D,g}|\mathrm{Imm}_D(X).$$

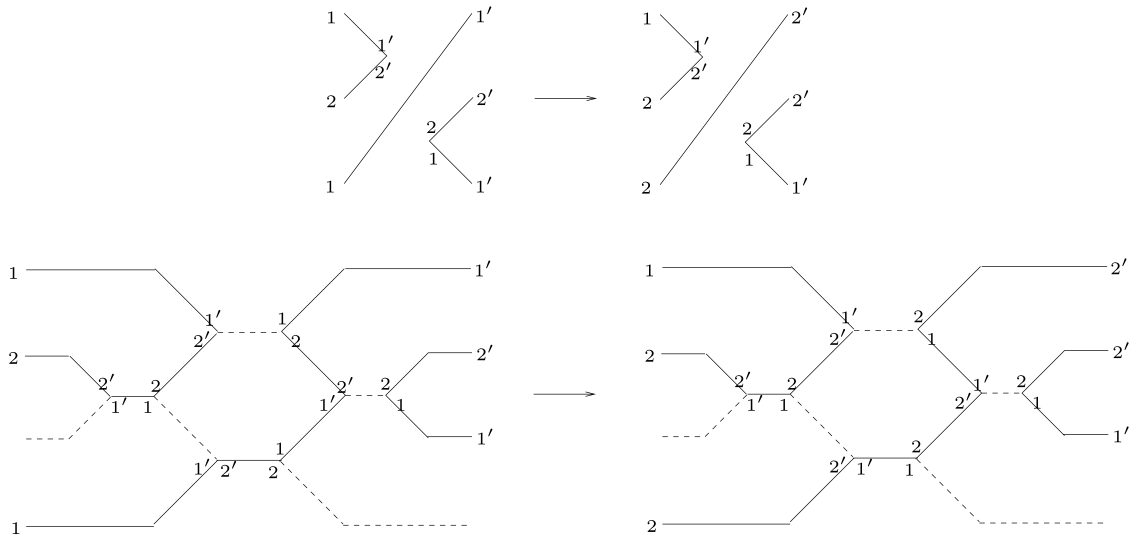

Fig. 13.

However, by definition $|L_{D,g}| = \sum_w |M_{w,g}||L_{D,g,w}|$. Then, we get

$$\sum_D |L_{D,g}|\mathrm{Imm}_D(X) = \sum_D \sum_w |M_{w,g}||L_{D,g,w}|\mathrm{Imm}_D(X).$$

However, $|M_{w,g}||L_{D,g,w}| = |M_{w,g}||L_{D,\tilde{g},w}|$ since if $M_{w,g}$ is non-empty we argued above that $|L_{D,g,w}| = |L_{D,\tilde{g},w}|$ and otherwise both sides are 0. Therefore,

$$\sum_D \sum_w |M_{w,g}||L_{D,g,w}|\mathrm{Imm}_D(X) = \sum_w |M_{w,g}| \sum_D |L_{D,\tilde{g},w}|\mathrm{Imm}_D(X) = \sum_w |M_{w,g}|\mathrm{Imm}^a_w(X).$$

On the other hand, from Theorem 6.1 we know that

$$\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X)\Delta_{I_3,J_3}(X) = \sum_w |M_{w,g}|\mathrm{Imm}^{TL}_w(X')\Delta_{I_3,J_3}(X).$$

Note that the number of different $\mathrm{Imm}^{TL}_w(X')\Delta_{I_3,J_3}(X)$-s, or in other words the number of w-s is the Catalan number and is equal to the dimension of the vector space generated by $\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X)\Delta_{I_3,J_3}(X)$-s with fixed (I_3, J_3), cf. [3]. Choose a subset of products $\Delta_{I_1,J_1}(X)\Delta_{I_2,J_2}(X)\Delta_{I_3,J_3}(X)$ that forms a basis of this vector space. We have just seen that transition matrices to this basis from $\mathrm{Imm}^{TL}_w(X')\Delta_{I_3,J_3}(X)$-s and $\mathrm{Imm}^a_w(X)$-s coincide. Then, their inverses exist and also coincide, from which we conclude that $\mathrm{Imm}^a_w(X) = \mathrm{Imm}^{TL}_w(X')\Delta_{I_3,J_3}(X)$. Thus, $a^D_{w,I_3,J_3} = |L_{D,\tilde{g},w}|$ as desired. □

7. Weighted networks

Let $G = (V, E)$ be a finite oriented acyclic planar graph with n sources followed by n sinks on the boundary of a Jordan curve. Let $\omega : E \longrightarrow R$ be a weight function assigning to each edge $e \in E$ the weight $\omega(e)$ in some commutative ring R. We refer to $N = (G, \omega)$ as a *weighted network*. A *path* p in N is a path from a source to a sink, and we define $\omega(p) = \prod_{e\in p} \omega(e)$. Let $P(N)$ be the set of all paths in N.

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a family of paths in $P(N)$ such that no four paths in $\mathbf{p}$ intersect in the same vertex, but without other restrictions. We denote $\omega(\mathbf{p}) = \prod \omega(p_i)$. Removing all edges in N which do not lie in any p_i, and marking as double or triple the edges used twice or thrice by $\mathbf{p}$, we get an underlying *marked subnetwork* $\tilde{N}(\mathbf{p})$ of N. We denote by $\tilde{N} < N$, the fact that $\tilde{N}$ is a marked subnetwork of N, and we denote by $P(\tilde{N})$ the set of all $\mathbf{p}$ such that $\tilde{N} = \tilde{N}(\mathbf{p})$.

Define a *vertical uncrossing* of a crossing of two or three paths by a procedure shown in Fig. 14.

Define $D(\tilde{N})$ to be the graph obtained by vertically uncrossing all the crossings in $\tilde{N}$. Then, it is clear that $D(\tilde{N}) \in \mathfrak{M}_n$ is a (possibly reducible) web. Let $e_{D(\tilde{N})} = \sum c_{i,\tilde{N}} e_{D_i}$ be the decomposition into reduced webs.

Let $\mathbf{I} = (I_1, I_2, I_3)$ and $\mathbf{J} = (J_1, J_2, J_3)$ be disjoint partitions of $[n]$ such that $|I_k| = |J_k|$. Let $P_{\mathbf{I},\mathbf{J}}(N)$ be the set of families $\mathbf{p}$ such that paths which start in I_k end in J_k, and the paths which start in the same I_k do not intersect.

Let $X(N)$ be the matrix given by $x_{i,j} = \sum \omega(p)$, where the sum is taken over all p starting at ith source and ending at jth sink. The following statement is known as *Lindström's lemma*, cf. [5].

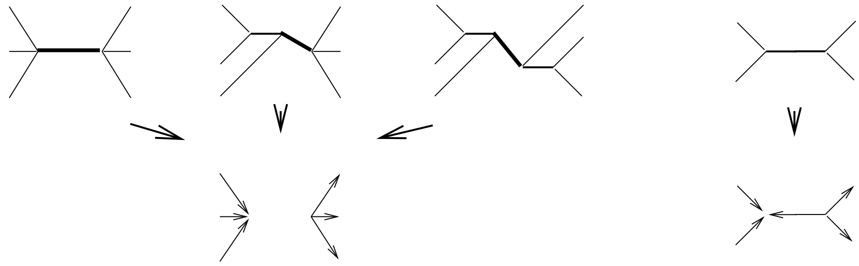

Fig. 14.

Lemma 7.1. *The determinant* $\Delta(X(N))$ *is equal to* $\sum_{\mathbf{p}} \omega(\mathbf{p})$, *where the sum is taken over all pairwise non-intersecting families of paths* $\mathbf{p}$ *in* $P(N)$.

Let

$$\mathrm{Imm}'_{D_i}(N) = \sum_{\tilde{N}<N} c_{i,\tilde{N}}\,\omega(\tilde{N}).$$

Let g be the boundary labeling determined by $(\mathbf{I}, \mathbf{J})$ as before. The following theorem is similar to [13, Proposition 26].

Theorem 7.2. *We have*

$$\Delta_{I_1,J_1}(X(N))\Delta_{I_2,J_2}(X(N))\Delta_{I_3,J_3}(X(N)) = \sum_i |L_{D_i,g}|\mathrm{Imm}'_{D_i}(N).$$

Proof. It is clear from the Lindström's lemma that

$$\Delta_{I_1,J_1}(X(N))\Delta_{I_2,J_2}(X(N))\Delta_{I_3,J_3}(X(N)) = \sum_{\mathbf{p}\in P_{\mathbf{I},\mathbf{J}}(N)} \omega(\mathbf{p}).$$

Note that the sum on the right involves only families $\mathbf{p}$ with no four paths crossing in one point. Label each path with k-s and k'-s if it starts at I_k. Then, the induced labeling of $D(\tilde{N}(\mathbf{p}))$ is consistent labeling, and furthermore this map is a bijection between $\bigcup_{\tilde{N}<N} L_{D(\tilde{N}),g}$ and $P_{\mathbf{I},\mathbf{J}}(N)$. Thus,

$$\sum_{\mathbf{p}\in P_{\mathbf{I},\mathbf{J}}(N)} \omega(\mathbf{p}) = \sum_{\tilde{N}<N} |L_{D(\tilde{N}),g}|\omega(\tilde{N}).$$

Recall from the proof of Theorem 3.4 that

$$|L_{D(\tilde{N}),g}| = \sum_{D_i} |L_{D(\tilde{N}),D_i,g}| = \sum_{D_i} c_{i,\tilde{N}}|L_{D_i,g}|.$$

Then

$$\begin{aligned}\sum_{\tilde{N}<N} |L_{D(\tilde{N}),g}|\omega(\tilde{N}) &= \sum_{\tilde{N}<N}\left(\sum_{D_i} c_{i,\tilde{N}}|L_{D_i,g}|\omega(\tilde{N})\right)\\ &= \sum_{D_i}\left(|L_{D_i,g}|\sum_{\tilde{N}<N} c_{i,\tilde{N}}\omega(\tilde{N})\right) = \sum_{D_i}|L_{D_i,g}|\mathrm{Imm}'_{D_i}(N). \quad \square\end{aligned}$$

Corollary 7.3. *We have* $\mathrm{Imm}_D(X(N)) = \mathrm{Imm}'_D(N)$.

Proof. The products of the complementary minors labeled by the *standard bitableaux* of [3] with at most three columns form a linear basis, the *standard basis*, for the subspace of immanants generated by products of triples of complementary minors. On the other hand we know that the number of those is exactly the dimension of TLM_n^3, i.e., the number of reduced webs in $\mathfrak{M}_n$. Therefore, the transition matrix from the Imm_D to the standard basis is invertible. Then, both the Imm_D and the Imm'_D are recovered via the same transition matrix from the standard basis, and thus they must coincide. $\square$

Note that this provides an alternative proof of Theorem 4.2 since by a result of Brenti [1] every totally nonnegative matrix can be represented by a planar weighted network with nonnegative weights. In fact we have implicitly used the result of Brenti in the original proof of Theorem 4.2 as well, when we relied on Lemma 4.3.

8. Concluding remarks

In [17] Rhoades and Skandera introduced a family of immanants called *Kazhdan–Lusztig immanants*, where the coefficients of monomials are given by evaluations of Kazhdan–Lusztig polynomials. Kazhdan–Lusztig immanants are labeled by permutations, and constitute a basis for the whole space of immanants. In [17] it is shown, relying on the work of Fan and Green [4], that Temperley–Lieb immanants coincide with the Kazhdan–Lusztig immanants for $(3, 2, 1)$-avoiding permutations. According to [18, Theorem 2.4] the Kazhdan–Lusztig immanants labeled by $(k, \dots, 1)$-avoiding permutations constitute a basis for the vector space generated by products of k-tuples of complementary minors. Thus, one is naturally led to wonder what is the relation between A_2-web immanants and $(4, 3, 2, 1)$-avoiding Kazhdan–Lusztig immanants. This question might be related to the question addressed in [8].

A theme of Schur positivity appears in the study of immanants, cf. [22,7,17,12]. In the terminology of [7,17] a *generalized Jacobi–Trudi matrix* corresponding to two partitions λ, μ is the matrix with entries $x_{i,j} = h_{\lambda_i - \mu_j}$, where the h are the *complete homogeneous symmetric functions*, cf. [20]. It was shown in [17], relying on a result of Haiman [7], that Kazhdan–Lusztig immanants of generalized Jacobi–Trudi matrices are nonnegative when expressed in the basis of *Schur functions*. One might wonder if web immanants have the same property. Note that if web immanants were shown to be nonnegative combinations of Kazhdan–Lusztig immanants, the Schur positivity would follow.

It is natural to expect a generalization of the present results from TLM_n^3 to TLM_n^k for any k. That would involve having a Kauffman diagram-like calculus for any k, which is essentially equivalent to the question of describing higher rank spiders. Progress has been made in this direction [9,15] but the question remains open. Note that the confluence property of Kuperberg's reduction rules is crucial for our construction since it allows us to have a set of reduced webs to which we associate the immanants.

Acknowledgements

The author would like to express gratitude to the following people: Thomas Lam for encouragement and helpful comments, Bruce Westbury for insightful comments on a draft, Richard Stanley for pointing out the bijection in the proof of Theorem 2.2, Mark Skandera and Greg Kuperberg for feedback on a draft, T. Kyle Petersen for his generous help with proofreading.

References

[1] F. Brenti, Combinatorics and total positivity, J. Combin. Theory Ser. A 71 (2) (1995) 175–218.
[2] T. Brzeziński, J. Katriel, Representation-theoretic derivation of the Temperley–Lieb–Martin algebras, J. Phys. A 28 (18) (1995) 5305–5312.
[3] J. Desarmenien, J. Kung, G.-C. Rota, Invariant theory, Young bitableaux, and combinatorics, Adv. Math. 27 (1) (1978) 63–92.
[4] K. Fan, R.M. Green, Monomials and Temperley–Lieb algebras, J. Algebra 190 (1997) 498–517.
[5] S. Fomin, A. Zelevinsky, Total positivity: tests and parametrizations, Math. Intelligencer 22 (1) (2000) 23–33.
[6] W. Fulton, Young Tableaux, Cambridge University Press, 1999.
[7] M. Haiman, Hecke algebra characters and immanant conjectures, J. Amer. Math. Soc. 6 (1993) 569–595.
[8] M. Khovanov, G. Kuperberg, Web bases for sl(3) are not dual canonical, Pacific J. Math. 188 (1) (1999) 129–153.
[9] D. Kim, Graphical calculus on representation of quantum Lie algebras, Ph.D. Thesis, UC Davis, 2003.
[10] G. Kuperberg, Spiders for rank 2 Lie algebras, Comm. Math. Phys. 180 (1) (1996) 109–151.
[11] G. Kuperberg, The quantum G_2 link invariant, Internat. J. Math. 5 (1) (1994) 61–85.
[12] T. Lam, A. Postnikov, P. Pylyavskyy, Schur positivity and Schur log-concavity, Amer. J. Math. (in press) arXiv:math.CO/0502446.
[13] T. Lam, P. Pylyavskyy, Temperley–Lieb pfaffinants and Schur Q-positivity conjectures, arXiv:math.CO/0612842.
[14] P. Martin, Potts Models and Related Problems in Statistical Mechanics, World Scientific, Singapore, 1991.
[15] S. Morrison, A diagrammatic category for the representation theory of $\mathfrak{U}_q(\mathfrak{sl}_n)$, Ph.D. Thesis, UC Berkeley, 2007.
[16] B. Rhoades, M. Skandera, Temperley–Lieb immanants, Ann. Comb. 9 (4) (2005) 451–494.
[17] B. Rhoades, M. Skandera, Kazhdan–Lusztig immanants and products of matrix minors, J. Algebra 304 (2006) 793–811.
[18] B. Rhoades, M. Skandera, Kazhdan–Lusztig immanants and products of matrix minors, II, Linear Multilinear Algebra (in press).
[19] A. Sikora, B. Westbury, Confluence theory for graphs, Algebr. Geom. Topol. 7 (2007) 439–478.
[20] R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge, 1999.
[21] J. Stembridge, Immanants of totally positive matrices are nonnegative, Bull. Lond. Math. Soc. 23 (1991) 422–428.
[22] J. Stembridge, Some conjectures for immanants, Canad. J. Math. 44 (5) (1992) 1079–1099.
[23] H.N.V. Temperley, E.H. Lieb, Proc. R. Soc. Lond. Ser. A 322 (251) (1971).
[24] B. Westbury, Invariant tensors for spin representations of $\mathfrak{so}(7)$, Math. Proc. Cambridge Philos. Soc. (in press).